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A new construction on the q -Bernoulli polynomials

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Abstract

This paper performs a further investigation on the q -Bernoulli polynomials and numbers given by Açıkgöz et al. (Adv. Differ. Equ. **2010**, 9, Article ID 951764) some incorrect properties are revised. It is pointed out that the definition concerning the q -Bernoulli polynomials and numbers is unreasonable. The purpose of this paper is to redefine the q -Bernoulli polynomials and numbers and correct its wrong properties and rebuild its theorems.

1 Introduction/Preliminaries

Many mathematicians have studied the q -Bernoulli, q -Euler polynomials and related topics (see [1-11]). It is worth that Açıkgöz et al. [1] give a new approach to the q -Bernoulli polynomials and the q -Bernstein polynomials and show some properties. That is, Açıkgöz et al. introduced a new generating function related the q -Bernoulli polynomials and gave a new construction of these polynomials related to the second kind Stirling numbers and the q -Bernstein polynomials in [1]. The purpose of this paper is to redefine a generating function related the q -Bernoulli polynomials and numbers and correct its wrong properties and rebuild its theorems.

In this paper, we assume that $q (\in \mathbb{C})$ is indeterminate with $|q| < 1$. The q -number is defined by $[x]_q = \frac{q^x - 1}{q - 1}$ (see [4-9]).

It is known that the Bernoulli polynomials are defined as

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{for } |t| < 2\pi \quad (1.1)$$

and that $B_n(0) = B_n$ are called the Bernoulli numbers.

The recurrence formula for the classical Bernoulli numbers B_n is as follows:

$$B_0 = 1 \text{ and } (B + 1)^n - B_n = 0 \quad \text{if } n > 0. \quad (1.2)$$

The q -extension of the following recurrence formula for the Bernoulli numbers is given by

$$B_{0,q} = 1 \text{ and } q(qB + 1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad (1.3)$$

with the usual convention of replacing B_q^n by $B_{n,q}$ (see [2,4]).

2 On the q -Bernoulli polynomials and numbers

In this section, we first recall the q -Bernoulli polynomials and numbers, then indicate the ambiguities on the Açıkgöz et al. [1]'s definition for the q -Bernoulli polynomials and redefine it. Counter-examples show that some properties are incorrect. Specially, these examples show that the concept on the generating function of the q -Bernoulli polynomials is unreasonable.

Definition 2.1 (Açıkgöz et al. [1]) For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q -Bernoulli polynomials as follows,

$$D_q(t, x) = -t \sum_{y=0}^{\infty} q^y e^{[x+y]_q t} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \quad (2.1)$$

Note that

$$\lim_{q \rightarrow 1} D_q(t, x) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{for } |t| < 2\pi, \quad (2.2)$$

where $B_n(x)$ are the classical Bernoulli polynomials.

In the special case $x = 0$, $B_{n,q}(0) = B_{n,q}$ are called the q -Bernoulli number.

That is,

$$D_q(t) = D_q(t, 0) = -t \sum_{y=0}^{\infty} q^y e^{[y]_q t} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \quad (2.3)$$

Remark 2.2 Definition 2.1 (Açıkgöz et al. [1]) is unreasonable, since it is not the generating functions of the q -Bernoulli polynomials and numbers. This can be seen the following counter-examples.

Counter-example 2.3 If we take $t = 0$ in (2.2) of Definition 2.1 (Açıkgöz et al. [1]), then we have $\lim_{q \rightarrow 1} D_q(0, x) = 0$. But $\lim_{t \rightarrow 0} \frac{t}{e^t - 1} e^{xt} = 1$ does not hold in the sense of Definition 2.1 (Açıkgöz et al. [1]).

Counter-example 2.4 From (2.1) of Definition 2.1 (Açıkgöz et al. [1]),

$$\begin{aligned} D_q(t, x) &= \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} \\ &= B_{0,q}(x) + \sum_{n=1}^{\infty} B_{n,q}(x) \frac{t^n}{n!}, \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} D_q(t, x) &= -t \sum_{y=0}^{\infty} q^y e^{[x+y]_q t} \\ &= -t \sum_{y=0}^{\infty} q^y \sum_{n=0}^{\infty} [x+y]_q^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(-\frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \sum_{y=0}^{\infty} q^{(l+1)y} \right) \frac{t^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} \left(-\frac{n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{lx} \frac{l}{1-q^{l+1}} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

Comparing these identities (2.4) and (2.5), we obtain

$$B_{0,q}(x) = 0 \text{ and } B_{n,q}(x) = -\frac{n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{lx} \frac{l}{1-q^{l+1}}. \quad (2.6)$$

This cannot satisfy some well-known results related the Bernoulli polynomials and numbers. For example, $B_0 = 1$.

Counter-example 2.5 From Definition 2.1 (Açikgöz et al. [1]), we note that

$$\begin{aligned} qD_q(t, 1) - D_q(t) &= -t \sum_{\gamma=0}^{\infty} q^{\gamma+1} e^{[1+\gamma]_q t} - t \sum_{\gamma=0}^{\infty} q^{\gamma} e^{[\gamma]_q t} \\ &= t, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} qD_q(t, 1) - D_q(t) &= q \sum_{n=0}^{\infty} B_{n,q}(1) \frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (qB_{n,q}(1) - B_{n,q}) \frac{t^n}{n!}. \end{aligned} \quad (2.8)$$

From (2.7) and (2.8), we can easily derive that

$$B_{n,q} = 0 \text{ and } qB_{n,q}(1) - B_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}. \quad (2.9)$$

From (2.1) of Definition 2.1 (Açikgöz et al. [1]),

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} &= D_q(t, x) \\ &= -t \sum_{\gamma=0}^{\infty} q^{\gamma} e^{[x+\gamma]_q t} \\ &= e^{[x]_q t} \frac{1}{q^x} D_q(tq^x) \\ &= \left(\sum_{l=0}^{\infty} \frac{[x]_q^l t^l}{l!} \right) \times \left(\sum_{m=0}^{\infty} B_{m,q} \frac{q^{(m-1)x} t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} B_{m,q} q^{(m-1)x} [x]_q^{n-m} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.10)$$

If we compare the coefficients on the both sides in (2.10),

$$B_{n,q}(x) = \sum_{m=0}^n \binom{n}{m} B_{m,q} q^{(m-1)x} [x]_q^{n-m}. \quad (2.11)$$

From (2.9) and (2.11),

$$B_{0,q}(x) = \frac{1}{q^x} B_{0,q} = 0. \quad (2.12)$$

However, these are also incorrect.

Next, we redefine the q -Bernoulli polynomials and numbers.

Definition 2.6 For $q \in \mathbb{C}$ with $|q| < 1$, let us define the q -Bernoulli polynomials $B_{n,q}(x)$ as follows,

$$F_q(t, x) = \frac{q-1}{\log q} e^{\frac{1}{1-q}t} - t \sum_{m=0}^{\infty} q^{x+m} e^{[x+m]_q t} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \quad (2.13)$$

Note that

$$\lim_{q \rightarrow 1} F_q(t, x) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \text{ for } |t| < 2\pi, \quad (2.14)$$

where $B_n(x)$ are the classical Bernoulli polynomials.

In the special case $x = 0$, $B_{n,q}(0) = B_{n,q}$ are called the q -Bernoulli numbers. That is,

$$F_q(t) = F_q(t, 0) = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \quad (2.15)$$

By simple calculations, we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} &= F_q(t, x) \\ &= e^{[x]_q t} F_q(q^x t) \\ &= \left(\sum_{m=0}^{\infty} \frac{[x]_q^m t^m}{m!} \right) \times \left(\sum_{l=0}^{\infty} B_{l,q} \frac{q^{lx} t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} B_{l,q} q^{lx} [x]_q^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.16)$$

Comparing the coefficients on the both sides in (2.16), we obtain

$$B_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} B_{l,q} q^{lx} [x]_q^{n-l}. \quad (2.17)$$

From (2.13) and (2.15), we derive the following equation.

$$B_{0,q} = \frac{q-1}{\log q} \text{ and } B_{n,q}(1) - B_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}. \quad (2.18)$$

By (2.17) and (2.18), we can see that

$$B_{0,q} = \frac{q-1}{\log q} \text{ and } \sum_{l=0}^n \binom{n}{l} B_{l,q} q^l - B_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}. \quad (2.19)$$

Theorem 2.7* For $n \in \mathbb{N}^*$, we have

$$B_{0,q} = \frac{q-1}{\log q} \text{ and } (qB_q + 1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}. \quad (2.20)$$

with the usual convention of replacing B_q^n by $B_{n,q}$.

Remark 2.8 Theorem 2.7* is a revised theorem of Theorem 2.1 in [1].

From (2.13), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} &= F_q(t, x) \\
 &= \frac{q-1}{\log q} e^{\frac{1}{1-q}t} - t \sum_{m=0}^{\infty} q^{x+m} e^{[x+m]_q t} \\
 &= \frac{q-1}{\log q} \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \frac{t^n}{n!} - \sum_{m=0}^{\infty} q^{x+m} \sum_{n=0}^{\infty} n[x+m]_q^{n-1} \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\frac{q-1}{\log q} \frac{1}{(1-q)^n} - n \sum_{m=0}^{\infty} q^{x+m} [x+m]_q^{n-1} \right) \frac{t^n}{n!} \quad (2.21) \\
 &= \sum_{n=0}^{\infty} \left(-\frac{(1-q)^n}{\log q} - \frac{n}{(1-q)^{n-1}} \sum_{m=0}^{\infty} q^{x+m} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{(x+m)l} \right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\frac{(q-1)^{1-n}}{\log q} + \frac{n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{l+1} q^{(l+1)x} \frac{1}{1-q^{(l+1)}} \right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{l}{[l]_q} \right) \frac{t^n}{n!}.
 \end{aligned}$$

By (2.21), we obtain the following theorem.

Theorem 2.9* For $n \in \mathbb{N}^*$, we have

$$B_{0,q} = \frac{q-1}{\log q} \text{ and } B_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lx} \frac{l}{[l]_q}. \quad (2.22)$$

Remark 2.10 Theorem 2.9* is a revised theorem of Theorem 2.3 in [1].

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Authors' contributions

Corresponding author raised the problem and make a sequence to approach the problem. AB carried out the q -Bernoulli polynomials studies, participated in the making new construction of the q -Bernoulli numbers. EJM carried out the calculation of [1]. JHJ participated in the sequence alignment. SJL performed the correction problem. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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